J. Fluid Mech. (1965), *vol.* 23, *part* 4, *pp.* 715–735
Printed in Great Britain

On the stability of the asymptotic suction boundary-layer profile

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(Received 18 January 1965)

This paper presents a discussion of some aspects of the linear stability problem for the asymptotic suction profile. An exact solution of the inviscid equation is first obtained in terms of the usual hypergeometric function and its analytical continuation. This exact solution provides both a corrected version of an earlier treatment by Freeman and an independent check on the more general method suggested for solving the inviscid equation numerically. Various approximations to the characteristic equation, and hence to the curve of neutral stability, are then considered. In particular, it is found that, in a consistent asymptotic treatment of the related adjoint problem, at least one viscous correction to the singular inviscid solution must be considered. Based on the present results for the adjoint problem, it is suggested that Tollmien's original treatment of the viscous corrections must be slightly modified.

1. Introduction

In the study of the stability of laminar boundary layers, the well-known asymptotic suction profile has played an important role. Because of its simple analytical form, this flow permits a more precise discussion of its stability properties than most other boundary-layer flows for which the velocity distribution is usually defined only numerically. The stability of this flow was first studied by Pretsch (1942) and by Bussmann & Münz (1942), and their results were later reviewed and extended by Freeman (Chiarulli & Freeman 1948). In particular, Freeman showed that the inviscid form of the Orr–Sommerfeld equation can be transformed into the usual hypergeometric equation and thus solved exactly. In spite of the fact that this is the only flow for which such an exact solution has been obtained, it is surprising that its importance has not been more widely recognized. Even Freeman's solution for the curve of neutral stability, based on this exact solution of the inviscid equation, though widely quoted (see, for example, Lin 1955 or Stuart 1963) has never been published.

The importance of this exact solution lies primarily in the understanding it provides of the analytical structure of the inviscid equation for the whole class of flows of the boundary-layer type. In general, however, the required solutions of the inviscid equation can only be obtained by some approximate method, one of the most powerful of which would appear to be the use of high-

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speed, numerical computation. Recently, in connexion with the development of such numerical methods, a new calculation of the neutral curve for the asymptotic suction profile was made but the results so obtained were found to be in substantial disagreement with Freeman's results in the neighbourhood of the minimum critical Reynolds number. Similar calculations for plane Poiseuille flow had previously agreed so well with Lin's (1945) results that the discrepancy found in this case was completely unexpected and too large to be easily explained. In §2, therefore, we present a re-examination of Freeman's solution which shows that an error had been made in the required analytical continuation of the solution. A numerical method of solving the inviscid equation, that is both simple and effective, is described briefly in §3, and it is then shown in §4 that complete agreement is obtained between the analytical and numerical solutions.

To complete the discussion of the neutral curve, it is necessary to obtain its asymptotic behaviour for large values of the Reynolds number. For this purpose one must first obtain the behaviour of a particular linear combination of the inviscid solutions for small values of the wave-number. The required result is easily obtained, as shown in §4, by first transforming the inviscid equation into a first-order Riccati equation, the solution of which can then be expanded in powers of the wave-number. This method of dealing with the inviscid part of the characteristic equation is applicable not only to flows of the boundary-layer type but also, with minor modifications, to other classes of flows. In addition, it provides a further check on the analytical and numerical solutions obtained in §§2 and 3.

The exact solution of the inviscid equation obtained in §2 can also be usefully employed to study a number of further aspects of the stability problem for such flows. Thus, for example, in §5 we consider the related adjoint formulation of the problem. The solutions of the adjoint Orr–Sommerfeld equation play an important role not only in the familiar initial-value problem (Schensted 1960) but also in the solution of the inhomogeneous Orr–Sommerfeld equation (Michael 1964) and in the non-linear stability theory for parallel flows (Stuart 1960; Watson 1960). On an exact basis, the solution of the adjoint problem must lead to exactly the same neutral curve since the eigenvalues of the two problems are the same. But when asymptotic approximations to the eigenfunctions of the two problems are used, it is found (Reid 1965) that the characteristic equations and hence the neutral curves are not identical, even at a comparable level of approximation. It is important, therefore, that detailed calculations be made in at least one case so as to be able to assess the differences between the two approaches.

In this discussion of the adjoint problem, it is found that at least one viscous correction to the singular inviscid solution must be used. As is well known, the precise manner in which this viscous correction is introduced is a matter of some delicacy. This question is therefore examined in further detail in §5 where it is suggested that, in a consistent treatment of these viscous corrections, Tollmien's (1929) original scheme must be slightly modified. The required modifications make little difference so far as the usual problem is concerned but they are clearly essential for the adjoint problem.

2. The analytical solution of the inviscid equation

In discussing the stability of the asymptotic suction profile, it is convenient to work exclusively in terms of non-dimensional quantities. For this purpose we introduce the free-stream velocity U_* as a characteristic velocity and the displacement thickness of the boundary layer L_* as a characteristic length ($L_* = -\nu_*/V_*$, where V_* is the suction velocity with $V_* < 0$ and constant). The velocity components of the basic flow can then be written in the non-dimensional form

$$U = 1 - e^{-y} \quad \text{and} \quad V = -1/R, \tag{2.1}$$

where R is the Reynolds number based on U_* and L_*. This basic flow is, of course, not 'strictly parallel' but, since $\partial U/\partial x \equiv 0$ and V is constant, the linearized disturbance equation that governs its stability can be derived with no further approximations beyond the usual linearization. Thus, if the disturbance stream function is taken in the usual form $\phi(y)\, e^{i\alpha(x-ct)}$, then an easy calculation shows that ϕ satisfies the modified Orr–Sommerfeld equation

$$(D^2-\alpha^2)^2\phi + (D^2-\alpha^2)\,D\phi = i\alpha R\{(U-c)\,(D^2-\alpha^2)\phi - U''\phi\}, \tag{2.2}$$

where $D = d/dy$ and we have already substituted for V from equation (2.1). On an exact basis, therefore, the stability of the asymptotic suction profile is governed not by the Orr–Sommerfeld equation but by the modified equation (2.2). The boundary conditions that must be satisfied by ϕ are

$$\phi = \phi' = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad \phi, \phi' \to 0 \quad \text{as} \quad y \to +\infty, \tag{2.3}$$

and we require, therefore, approximations to the two solutions of equation (2.2) that remain bounded as $y \to +\infty$.

It may be noticed that the inviscid form of equation (2.2) is identical with the inviscid form of the Orr–Sommerfeld equation and, at the usual level of asymptotic approximation, one of the required solutions is simply the solution of the inviscid equation that remains bounded as $y \to +\infty$. We shall denote this solution by $\Phi(y)$ and, to be definite, require that it be normalized by the condition $\Phi(y_c) = 1$, where y_c is the point where $U - c = 0$. When considered as an asymptotic approximation to one of the bounded solutions of either equation (2.2) or the Orr–Sommerfeld equation, however, this solution provides a valid approximation only in the region of the complex y-plane for which $-\frac{7}{6}\pi < \arg(y-y_c) < \frac{1}{6}\pi$ excluding the immediate neighbourhood of the critical point. The behaviour of $\Phi(y)$ in the 'viscous sector' $\frac{1}{6}\pi < \arg(y-y_c) < \frac{5}{6}\pi$ is discussed in further detail in § 5 below.

Consider then the inviscid form of equation (2.2) with $U(y)$ given by equation (2.1)

$$(1-e^{-y}-c)\,(\phi''-\alpha^2\phi) + e^{-y}\phi = 0. \tag{2.4}$$

This equation has a regular singular point at $y = y_c$, where $y_c = -\log(1-c)$, and an irregular singular point at $y = \infty$. As Freeman has shown, however, the transformation

$$\phi(y) = \exp\{-\alpha(y-y_c)\}f(t), \quad \text{where} \quad t = e^{-y}/(1-c), \tag{2.5}$$

leads to the equation

$$t(1-t)f'' + (1+2\alpha)\,(1-t)f' + f = 0, \tag{2.6}$$

which is of hypergeometric type. This equation has three singular points at $t = 0$, 1, and ∞ but they are now all regular. The transformation $t = e^{-y}/(1-c)$ maps the point $y = 0$ into the point $t = t_0 = 1/(1-c) > 1$ for $0 < c < 1$, the point $y = y_c$ into the point $t = t_c = 1$, and the point $y = +\infty$ into the point $t = 0$.

We thus require the solution of equation (2.6) that is regular in the neighbourhood of $t = 0$. This solution must clearly be a constant multiple of the usual hypergeometric function

$$F(p,q;r;t) = \frac{\Gamma(r)}{\Gamma(p)\,\Gamma(q)} \sum_{n=0}^{\infty} \frac{\Gamma(p+n)\,\Gamma(q+n)}{\Gamma(r+n)} \frac{t^n}{n!}, \tag{2.7}$$

where $$p = \alpha + (1+\alpha^2)^{\frac{1}{2}}, \quad q = \alpha - (1+\alpha^2)^{\frac{1}{2}}, \quad \text{and} \quad r = 1+2\alpha. \tag{2.8}$$

Since (for α real) $p+q-r = -1 < 0$, the series (2.7) is absolutely convergent for $|t| \leqslant 1$ and, to satisfy the prescribed normalization condition, we therefore take the solution in the form

$$f(t) = \frac{F(p,q;r;t)}{F(p,q;r;1)} \quad \text{for} \quad |t| \leqslant 1, \tag{2.9}$$

where, by the Gauss formula,

$$F(p,q;\, r;\, 1) = -\frac{\Gamma(r)}{\Gamma(p)\,\Gamma(q)}. \tag{2.10}$$

Since it will later be necessary to evaluate f and f' at $t = t_0 > 1$, it is necessary to obtain the analytical continuation of (2.9) into the region $|t| > 1$. The hypergeometric function (2.7) has a branch point at $t = 1$ and, in the usual treatments of this function, a cut is made in the t-plane from 1 to ∞ along the positive real axis. Since this inviscid solution provides a valid asymptotic approximation only in the sector $-\frac{7}{6}\pi < \arg(y-y_c) < \frac{1}{6}\pi$ of the complex y-plane, we must choose a path in the y-plane running from 0 to $+\infty$ that lies *below* y_c. Such a path in the y-plane corresponds to a path in the t-plane that runs from t_0 to 0 and lies *above* $t_c = 1$. We shall suppose therefore that the branch cut from 1 to ∞ in the t-plane lies *below* the real axis. In the present problem, since $p+q-r$ is a negative integer, the required analytical continuation contains logarithmic terms and can be written in the form (see, for example, Erdélyi, Magnus, Oberhettinger & Tricomi 1953)

$$f(t) = 1-(1-t)\,F(p+1,q+1;2;1-t)\log(1-t) - \sum_{n=0}^{\infty} A_n \frac{(1-t)^{n+1}}{(n+1)!}, \tag{2.11}$$

where

$$A_n(\alpha) = \frac{\Gamma(p+1+n)\,\Gamma(q+1+n)}{\Gamma(p+1)\,\Gamma(q+1)\,\Gamma(n+1)} \times \{\psi(p+1+n)+\psi(q+1+n)-\psi(n+1)-\psi(n+2)\} \tag{2.12}$$

and $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function. This form of the solution is valid in the region $|1-t| < 1$ of the cut t-plane and thus gives the required continuation provided $c < \frac{1}{2}$, i.e. $t_0 < 2$. In Freeman's version of equation (2.11) some additional terms were erroneously included; these terms are, in fact, formally zero but this was not recognized and led to incorrect results at a later stage in his calculations.

For the asymptotic suction profile, it can easily be verified *a posteriori* that the values of c along the neutral curve are nowhere large and this fact suggests that the values of f and f' at $t = t_0$ can conveniently be obtained by expansions in powers of c. The final expressions are substantially simpler, however, if one uses the small parameter $\epsilon = c/(1-c)$ rather than c itself. In keeping with a path of integration that lies above the critical point $t_c = 1$, if $\log(1-t) = \log|1-t|$ for $t < 1$ and real, then $\log(1-t)$ must be taken as $\log|1-t| - \pi i$ for $t > 1$ and real. Thus, in particular, we have $\log(1-t_0) = \log\epsilon - \pi i$. The required expansions for $f(t_0)$ and $f'(t_0)$ are then found to be

$$f(t_0) = 1 + (\epsilon - \alpha\epsilon^2)(\log\epsilon - \pi i) + h_0(\alpha)\,\epsilon - h_1(\alpha)\,\epsilon^2 + O(\epsilon^3 \log\epsilon) \tag{2.13}$$

and

$$\begin{aligned} f'(t_0) = {} & (1 + \log\epsilon - \pi i)(1 - \alpha\epsilon + \tfrac{1}{2}\alpha\epsilon^2 + \tfrac{2}{3}\alpha^2\epsilon^2) - (\log\epsilon - \pi i)(\alpha\epsilon - \alpha\epsilon^2 - \tfrac{4}{3}\alpha^2\epsilon^2) \\ & + h_0(\alpha) - 2h_1(\alpha)\,\epsilon + 3h_2(\alpha)\,\epsilon^2 + O(\epsilon^3\log\epsilon), \end{aligned} \tag{2.14}$$

where

$$\left.\begin{aligned} h_0(\alpha) &= \psi(p+1) + \psi(q+1) - \psi(2) - \psi(1), \\ h_1(\alpha) &= \alpha[\psi(p+2) + \psi(q+2) - \psi(3) - \psi(2)], \\ \text{and}\quad h_2(\alpha) &= (\tfrac{1}{2}\alpha + \tfrac{2}{3}\alpha^2)\,[\psi(p+3) + \psi(q+3) - \psi(4) - \psi(3)]. \end{aligned}\right\} \tag{2.15}$$

The expressions for $f(t_0)$ and $f'(t_0)$ can easily be evaluated since only tables of the digamma function are required (see, for example, Davis 1933).

Since α and c are of the same order, and hence both small, Freeman has suggested the further approximation in which the coefficients (2.15) are also expanded in powers of α and only those terms in $f(t_0)$ and $f'(t_0)$ up to $\alpha^m\epsilon^n$ with $m + n = 3$ are retained. This approximation then leads to the simple formulas

$$f(t_0) \fallingdotseq 1 + (\epsilon - \alpha\epsilon^2)(\log\epsilon - \pi i) - (\epsilon/\alpha) - \tfrac{1}{2}\epsilon + T_1\alpha\epsilon + T_2\alpha^2\epsilon + \alpha\epsilon^2 \tag{2.16}$$

and

$$\begin{aligned} f'(t_0) \fallingdotseq {} & (1 + \log\epsilon - \pi i)(1 - \alpha\epsilon + \tfrac{1}{2}\alpha\epsilon^2) - (\alpha\epsilon - \alpha\epsilon^2)(\log\epsilon - \pi i) \\ & - (1/\alpha) - \tfrac{1}{2} + (\alpha - 2\alpha^2\epsilon)\,T_1 + \alpha^2 T_2 + \alpha^3 T_3 + 2\alpha\epsilon - \tfrac{3}{4}\alpha\epsilon^2, \end{aligned} \tag{2.17}$$

where

$$T_1 = \tfrac{1}{3}\pi^2 - \tfrac{5}{4}, \quad T_2 = \tfrac{1}{2} - 2\zeta(3), \quad T_3 = \tfrac{1}{45}\pi^4 + \tfrac{1}{16}, \tag{2.18}$$

and $\zeta(z)$ is the Riemann Zeta-function.

This solution of the inviscid equation is unusual in that the required analytical continuation can be obtained exactly and hence that the expansions (2.13) and (2.14) can easily be extended to arbitrarily high orders in ϵ. For other flows, for which such an exact solution cannot be obtained, it is necessary to resort to some approximate method. The numerical method described in the following section would appear to be particularly useful in such cases.

3. The numerical solution of the inviscid equation

In this section we will describe a numerical method of solving the inviscid equation to obtain the bounded solution that we have called $\Phi(y)$. Sufficiently near the critical point the solutions of the inviscid equation can be taken in the form first given by Tollmien (1929):

$$\phi_A(y) = (y - y_c)\,P_A(y - y_c), \tag{3.1}$$

$$\phi_B(y) = P_B(y - y_c) + (U_c''/U_c')\,\phi_A(y)\log(y - y_c), \tag{3.2}$$

where P_A and P_B are power series in $y-y_c$, the leading terms of which are unity. We shall also suppose that ϕ_B contains no multiple of ϕ_A, i.e. that the coefficient of $y-y_c$ in P_B is zero. The required solution Φ must, therefore, be a linear combination of ϕ_A and ϕ_B, and the requirement that $\Phi(y_c) = 1$ means that it must be of the form

$$\Phi(y) = A\phi_A(y) + \phi_B(y). \tag{3.3}$$

The constant A, which may, in general, depend on the parameters α and c, must be determined so that Φ remains bounded as $y \to +\infty$. This is the central difficulty in the numerical approach to the determination of Φ.

The inviscid equation for this problem has regular singular points at $y = y_c \pm 2m\pi i$ $(m = 0, 1, 2, \ldots)$ so that, in general, the power series P_A and P_B are convergent only for $|y-y_c| < 2\pi$ and the constant A cannot be determined therefore from the solutions (3.1) and (3.2) alone. By changing the independent variable to $s = e^{-y}$, however, a convenient representation for Φ can be obtained that is valid in the neighbourhood of $y = +\infty$. The inviscid equation then becomes

$$(U-c)(s^2\ddot{\phi} + s\dot{\phi} - \alpha^2\phi) - U''\phi = 0, \tag{3.4}$$

where a dot denotes differentiation with respect to s. The point $s = 0$ is now a regular singular point of equation (3.4) with exponents $\pm\alpha$; the solution of this equation that is bounded in the neighbourhood of $s = 0$ must therefore be of the form $s^\alpha P_\infty(s)$, where $P_\infty(s)$ is also a power series in s with a leading term of unity. For some purposes it is convenient to obtain $P_\infty(s)$ as the solution of the differential equation

$$(U-c)\{s^2\ddot{P}_\infty + s(1+2\alpha)\dot{P}_\infty\} - U''P_\infty = 0 \tag{3.5}$$

that satisfies the initial conditions

$$P_\infty(0) = 1, \quad \dot{P}_\infty(0) = -1/(1+2\alpha)(1-c). \tag{3.6}$$

Thus we have

$$\Phi(y) = Bs^\alpha P_\infty(s), \tag{3.7}$$

where B is a second constant that will also, in general, depend on both α and c. This representation for Φ is valid in the interval $0 \leqslant s < 1-c$ or $0 < y-y_c \leqslant +\infty$. For other velocity profiles, however, this representation may not hold over the whole of this range.

We thus have two overlapping representations for Φ, and the constants A and B appearing in equations (3.3) and (3.7) can now be determined by requiring that Φ and Φ' be continuous at some point in their common domain of validity. The matching point was chosen, for convenience, to be at $y = 1$, $s = 1/e$. Thus, A and B are determined by the pair of algebraic equations

$$A\phi_A(1) + \phi_B(1) = Be^{-\alpha}P_\infty(1/e)$$

and

$$A\phi'_A(1) + \phi'_B(1) = -Be^{-\alpha}\{\alpha P_\infty(1/e) + e^{-1}\dot{P}_\infty(1/e)\}. \tag{3.8}$$

The continuation of the solution (3.3) to the interval $y < y_c$, by a path lying below y_c, then provides the required values of Φ and Φ' at the boundary $y = 0$.

The computational procedures actually used to obtain the values of Φ and Φ' at $y = 0$ are similar to the ones suggested by Conte & Miles (1959) and, therefore, need only be described briefly. Numerical integration was used to obtain the values of ϕ_A and P_B (and hence ϕ_B) and their derivatives at $y = 0$ and 1. To obtain the required initial values, the power series for ϕ_A and P_B were first evaluated at the points $y_c^{\pm}$(say), where $y_c^+ > y_c$ and $y_c^- < y_c$. A fourth-order Runge-Kutta procedure was then used to integrate from y_c^+ to 1 and from y_c^- to 0. Since the coefficients in the power series P_A and P_B satisfy simple recursion relationships, the summation of these series by an electronic computer is not difficult. Furthermore, some preliminary experimentation showed that the terms in these series are rapidly decreasing even for $|y-y_c|$ as large as 0·25, and it is not necessary therefore to choose the initial points $y_c^{\pm}$ particularly close to y_c. In the case of P_∞ and $\dot{P}_\infty$, it was found that direct summation of the series was the best way to compute $P_\infty(1/e)$ and $\dot{P}_\infty(1/e)$.

In the case of the asymptotic suction profile, it is possible to obtain exact expressions for the constants A and B from the analytical solution given in § 2, and these results provide a further check on the purely numerical approach described above. The constant A, for example, is clearly the coefficient of $y-y_c$ in the expansion of the regular part of $\exp\{-\alpha(y-y_c)\}f(t)$, where $f(t)$ is given by equation (2.11). A simple calculation then gives

$$A(\alpha) = 1-2\gamma-\alpha-\psi(p+1)-\psi(q+1), \tag{3.9}$$

where $\gamma = 0{\cdot}5772\ldots$ is Euler's constant. For small values of α we have

$$A(\alpha) = \frac{1}{\alpha}+\tfrac{1}{2}-(\tfrac{1}{3}\pi^2-\tfrac{1}{4})\,\alpha+O(\alpha^2). \tag{3.10}$$

Similarly, from equations (2.9) and (2.10) we have

$$B(\alpha, c) = -\frac{\Gamma(p)\,\Gamma(q)}{\Gamma(r)}\exp(\alpha y_c). \tag{3.11}$$

Thus, in this case, A depends only on α, but B depends on both α and c.

4. The solution of the characteristic equation

The inviscid solution $\Phi(y)$ discussed in §§ 2 and 3 provides a valid asymptotic approximation to one of the bounded solutions of equation (2.2) in the region of the complex y-plane for which $-\frac{7}{6}\pi < \arg(y-y_c) < \frac{1}{6}\pi$ excluding the immediate neighbourhood of the critical point. This solution can be modified, however, to provide a valid approximation in the 'viscous sector' $\frac{1}{6}\pi < \arg(y-y_c) < \frac{5}{6}\pi$ including a neighbourhood of the critical point, but such refinements are not required in a first approximation to the characteristic equation. An asymptotic approximation to the second bounded solution of equation (2.2) is of a viscous type and will be taken, as usual, as the solution of the Airy equation

$$\phi^{\text{iv}} = i\alpha R U_c'(y-y_c)\,\phi''$$

that is exponentially small for $|y-y_c| \gg |\alpha R U_c'|^{-\frac{1}{3}}$ in a sector of the complex y-plane that includes the positive real axis.

In this approximation, the characteristic equation can be written in the form (see, for example, Lin 1955, p. 40)

$$\frac{w-1}{(1+\lambda)\,w} = F(z), \tag{4.1}$$

where
$$w(\alpha,c) = \left[1+\frac{U'(0)}{c}\frac{\Phi(0)}{\Phi'(0)}\right]^{-1}, \quad 1+\lambda(c) = \frac{U'(0)}{c}y_c, \tag{4.2}$$

and $F(z)$ is the Tietjens function with argument $z = (\alpha R U_c')^{\frac{1}{3}} y_c$. The complex parameter $w(\alpha, c)$ that appears in the characteristic equation (4.1) can be expressed in terms of the exact solution of §2 in the form

$$w(\alpha,c) = \frac{\alpha c + c t_0 f'(t_0)/f(t_0)}{-1+\alpha c + c t_0 f'(t_0)/f(t_0)}, \tag{4.3}$$

where $t_0 = 1/(1-c)$ and f is given by equation (2.11).

The left-hand side of equation (4.1) depends only on the solution of the inviscid equation and hence only on α and c, whereas the right-hand side depends only on z. The curve of neutral stability can therefore be obtained in a direct manner by means of the usual graphical procedure in which the real and imaginary parts of equation (4.1) are drawn on the same graph as shown in figure 1. The inviscid lines α = const. and c = const. were obtained by the numerical procedure

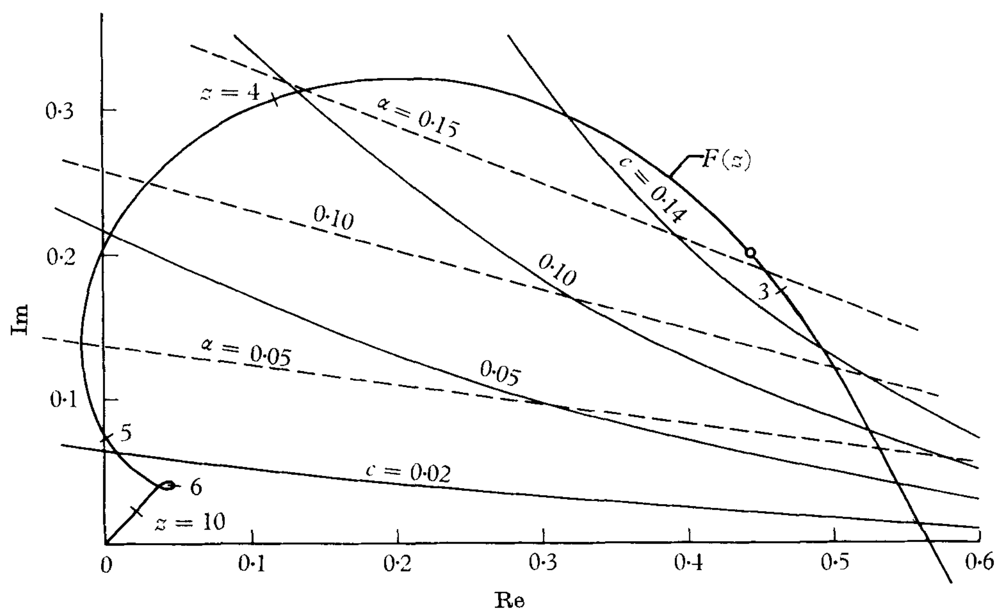

FIGURE 1. The graphical solution of the characteristic equation (4.1) for the asymptotic suction profile. The circled point corresponds to the minimum critical Reynolds number. The Tietjens function is from Miles (1960).

described in §3. The curve of neutral stability obtained in this way is shown as the solid curve in figure 2, and the corresponding behaviour of the wave speed c is shown in figure 3. It should be emphasized here that the kink on the upper

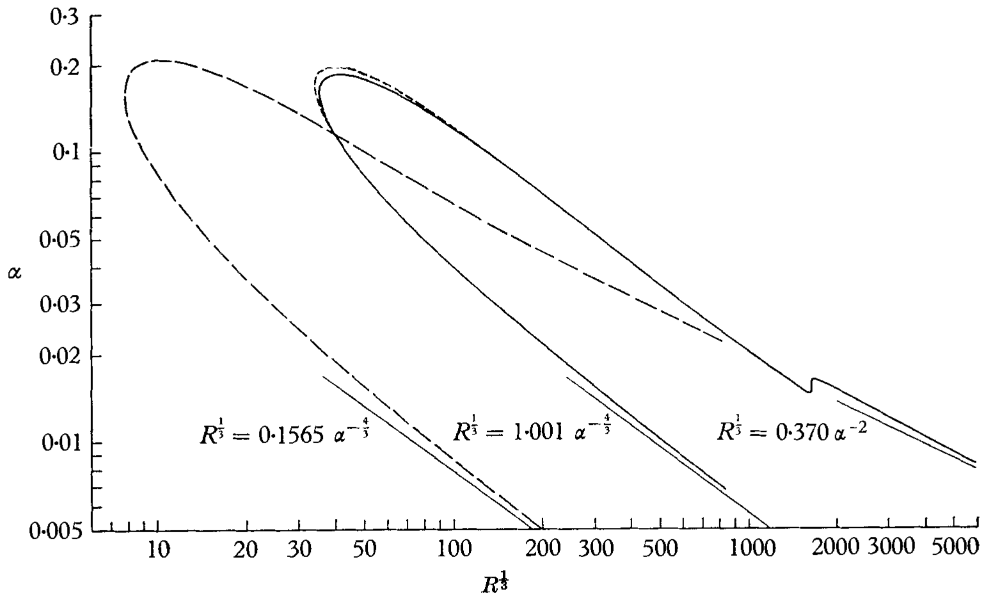

FIGURE 2. The curves of neutral stability for the asymptotic suction profile. The solid curve is based on equation (4.1); the long-dashed curve is based on equation (5.17); and the short-dashed curve is based on equation (5.24).

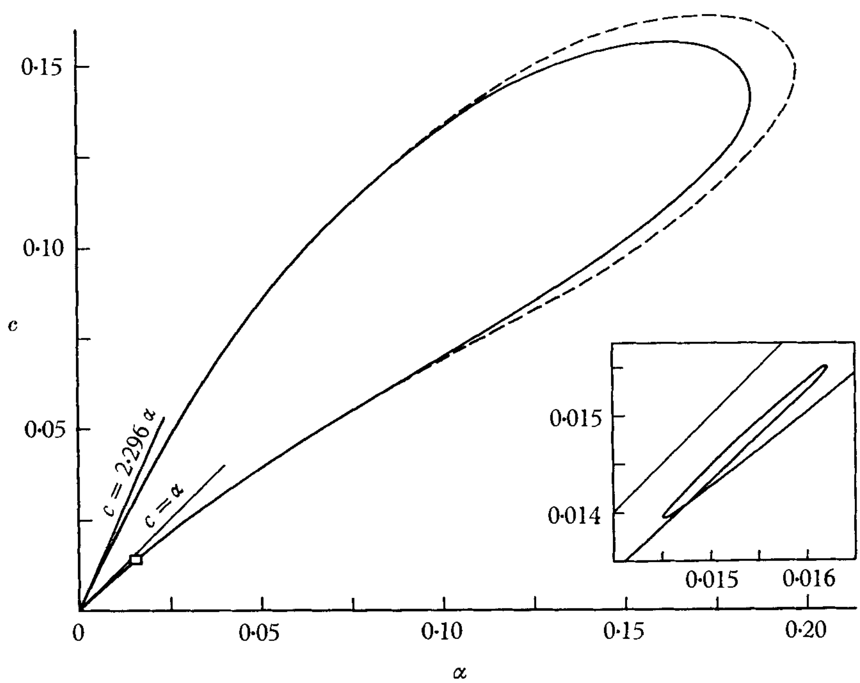

FIGURE 3. The relationship between the wave-number α and the wave-speed c along the neutral curves. The dashed curve is based on the adjoint characteristic equation (5.24).

branch of the neutral curve in figure 2 and the loop on the lower branch of the curve in figure 3 would not appear to have any physical significance but are merely a consequence of the asymptotic approximations that underlie the characteristic equation (4.1). This point is discussed further in § 6 where it is suggested that if suitable viscous corrections are applied to the singular part of the inviscid solution $\Phi(y)$, then both the kink and the loop would disappear.

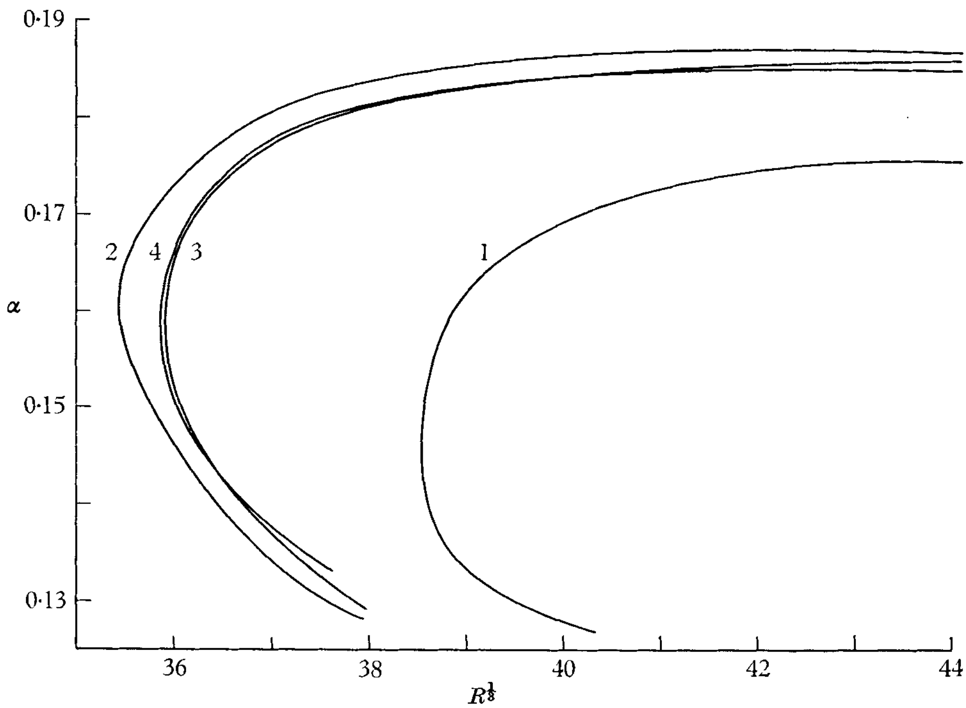

FIGURE 4. Approximations to the curve of neutral stability based on retaining terms up to $\alpha^m\epsilon^n$ in equations (2.16) and (2.17). Curve 1: $m+n = 2$; curve 2: $m+n = 3$; curve 3: $m+n = 4$; curve 4: $m+n = 5$. Curve 4 is also indistinguishable from the solution obtained by numerical integration.

When the results just described were first obtained, it was found that the value of the minimum critical Reynolds number differed from Freeman's result by about 9 %, and this discrepancy led to the re-examination and subsequent correction of Freeman's analytical solution described in § 2. As a further check on both the numerical and analytical solutions of the inviscid equation, the expansions (2.16) and (2.17) were used to provide an independent calculation of R_{cr} in the neighbourhood of R_{min}. The results of this calculation are given in figure 4, and they show that only a few terms need be retained in the expansions (2.16) and (2.17) to obtain essentially complete agreement with the numerical solution.

The behaviour of the inviscid solution for small values of α

To complete the discussion of the neutral curve, it is necessary to determine its asymptotic behaviour as $R \to \infty$. For this purpose it is first necessary to obtain the behaviour of the inviscid solution or, more precisely, of the ratio

$$\Phi'(0)/\Phi(0) \quad \text{as} \quad \alpha \to 0.$$

This has been done by Lin (1945) by a clever, but somewhat complicated, reordering of the terms in the Heisenberg type solutions of the inviscid equation. Furthermore, Lin's result is only valid (for flows of the boundary-layer type) if, with a suitably chosen length scale $U(y) \equiv 1$ for $1 \leqslant y \leqslant \infty$. Both of these difficulties are avoided in the present discussion in which the inviscid equation is first transformed into a first-order Riccati equation, the solution of which can then be expanded in powers of α. This procedure, in effect, provides the required analytical continuation of the inviscid solution termwise in α.

The inviscid equation can, of course, be reduced to a first-order equation in many different ways. For flows of the boundary-layer type, however, we find it convenient, following Miles (1962), to introduce the new dependent variable

$$\Omega(y) = \Phi/(U-c)\,[U'\Phi-(U-c)\,\Phi'], \tag{4.4}$$

which satisfies the Riccati equation

$$\Omega' = \alpha^2(U-c)^2\,\Omega^2-(U-c)^{-2}. \tag{4.5}$$

The inviscid parameter $w(\alpha, c)$ defined by equation (4.2) is then given by†

$$w(\alpha, c) = 1+cU'(0)\,\Omega(0). \tag{4.6}$$

Since $\Phi(y) \sim$ constant $e^{-\alpha y}$ as $y \to \infty$, we must have $\Phi'/\Phi \to -\alpha$ as $y \to \infty$. The required boundary condition on $\Omega(y)$ must therefore be

$$\Omega(y) \to 1/\alpha(1-c)^2 \quad \text{as} \quad y \to \infty, \tag{4.7}$$

where we have used the fact that $U(y) \to 1$ as $y \to \infty$. This result suggests that the solution of equation (4.5) can be expanded in the form

$$\Omega(y) = \frac{1}{\alpha(1-c)^2}+\sum_{n=0}^{\infty}\Omega_n(y)\,\alpha^n. \tag{4.8}$$

On substituting this expression into equation (4.5) and equating to zero the coefficients of like powers of α, we obtain the equations

$$\left.\begin{aligned}
\Omega_0' &= (1-c)^{-4}\,(U-c)^2-(U-c)^{-2},\\
\Omega_1' &= 2(1-c)^{-2}\,(U-c)^2\,\Omega_0,\\
\Omega_2' &= (1-c)^{-2}\,(U-c)^2\,[(1-c)^2\,\Omega_0^2+2\Omega_1],\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{4.9}$$

The solutions of these equations which satisfy the boundary conditions

$$\Omega_n(y) \to 0 \quad \text{as} \quad y \to \infty\,(n = 0, 1, 2, \ldots)$$

† If $U'(0) = 0$, then this procedure must be slightly modified (cf. Hughes & Reid 1965).

can all be obtained by a single quadrature in the form

$$\left.\begin{aligned}
\Omega_0(y) &= -(1-c)^{-2}\int_y^\infty \left[\left(\frac{U-c}{1-c}\right)^2 - \left(\frac{1-c}{U-c}\right)^2\right] dy,\\
\Omega_1(y) &= -2(1-c)^{-2}\int_y^\infty (U-c)^2\,\Omega_0(y)\,dy,\\
\Omega_2(y) &= -(1-c)^{-2}\int_y^\infty [(1-c)^2(U-c)^2\,\Omega_0^2(y) + 2(U-c)^2\,\Omega_1(y)]\,dy,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{4.10}$$

where the path of integration must lie below y_c.

If we now let $\Omega_n(0) = I_n(c)$ $(n = 0, 1, 2, \ldots)$, then from equation (4.6) we have

$$w(\alpha, c) = cU'(0)/\alpha(1-c)^2 + 1 + cU'(0)\,[I_0(c) + I_1(c)\,\alpha + I_2(c)\,\alpha^2 + O(\alpha^3)]. \tag{4.11}$$

It is useful to note that the imaginary part of equation (4.11) can be written explicitly in the form

$$v(\alpha, c) = -\pi c U'(0)\,\frac{U_c''}{U_c'^3}\left[1 - 2\alpha\int_0^{y_c}\left(\frac{U-c}{1-c}\right)^2 dy + O(\alpha^2)\right]. \tag{4.12}$$

Furthermore, for small values of c we have

$$\int_0^{y_c}\left(\frac{U-c}{1-c}\right)^2 dy = \frac{c^3}{3U'(0)} + O(c^4) \tag{4.13}$$

so that to a good approximation we can write

$$v(\alpha, c) \fallingdotseq -\pi c U'(0)\,U_c''/U_c'^3. \tag{4.14}$$

These results for small values of α are in agreement with Lin's (1945) results when suitably interpreted.

In the case of the asymptotic suction profile, all of the integrals $I_n(c)$ can be evaluated explicitly. For $n = 0$ and 1, for example, we have

$$I_0(c) = (1-c)^{-4}\left[-\frac{1}{c} + \tfrac{7}{2} - 3c + (1-c)^2\left(\log\frac{1-c}{c} + \pi i\right)\right] \tag{4.15}$$

and
$$\begin{aligned}
I_1(c) = (1-c)^{-6}\Big[&\tfrac{1}{4} + c - 4c^2 + 3c^3 - 2(1-c)^4 M\left(\frac{1}{1-c}\right)\\
&+ (1-c)^2(2c - 3c^2)\left(\log\frac{1-c}{c} + \pi i\right)\Big],
\end{aligned} \tag{4.16}$$

where
$$M\left(\frac{1}{1-c}\right) = -\mathrm{Li}_2(1-c) + \tfrac{1}{3}\pi^2 - \tfrac{1}{2}\log^2(1-c) - \pi i \log(1-c) \tag{4.17}$$

and $\mathrm{Li}_2(z)$ is the dilogarithm function defined by

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-z)}{z}\,dz. \tag{4.18}$$

For tables and properties of this function see, for example, Lewin (1958). The explicit form of $I_2(c)$ is also known but it is too lengthy to be recorded here. It is also interesting to observe, however, that if $I_0(c)$, $I_1(c)$, and $I_2(c)$ are further expanded in terms of the parameter $\epsilon = c/(1-c)$ as in §2, then (4.11) becomes

$$w(\alpha, c) \doteqdot (\epsilon+\epsilon^2)\,\alpha^{-1} + (-\epsilon\log\epsilon - \tfrac{1}{2}\epsilon - \epsilon^2\log\epsilon + \tfrac{3}{2}\epsilon^2 + \tfrac{1}{2}\epsilon^3) + (\tfrac{1}{4}\epsilon - \tfrac{1}{3}\pi^2\epsilon + \tfrac{1}{4}\epsilon^2 - \tfrac{1}{3}\pi^2\epsilon^2)\,\alpha - T_2\epsilon\alpha^2 + (\epsilon+\epsilon^2)\,\pi i \quad (4.19)$$

and this result, to this order, is identical with the hypergeometric solution of §2.

The asymptotic behaviour of the neutral curve can now be obtained by noting that, as $\alpha \to 0$, equation (4.1) becomes

$$F(z) = \frac{1}{1+\lambda}\left\{1 - \frac{\alpha}{cU'(0)}(1-c)^2 + \frac{\alpha^2}{cU'(0)}(1-c)^4\left[\frac{1}{cU'(0)} + I_0(c)\right] + O(\alpha^3)\right\}. \quad (4.20)$$

Since $\operatorname{Re}\{I_0(c)\} = -[cU'(0)]^{-1} + O(\log c)$ as $c \to 0$, it follows that

$$F_r(z) \to 1 - \alpha/cU'(0). \quad (4.21)$$

Thus we have (cf. figure 3)

$$c \sim \alpha/U'(0) \quad \text{and} \quad c \sim 2{\cdot}296\,\alpha/U'(0) \quad (4.22)$$

along the upper and lower branches of the neutral curve, respectively. For small values of c, equations (4.14) and (4.20) give

$$F_i(z) \to -\pi\frac{\alpha^2}{cU'(0)}\frac{U''(0)}{[U'(0)]^3} \quad (4.23)$$

from which we have (cf. figure 2)

$$R \sim \frac{1}{2\pi^2}\frac{[U'(0)]^{11}}{[U''(0)]^2}\alpha^{-6} \quad \text{along the upper branch} \quad (4.24)$$

and

$$R \sim 1{\cdot}001\,U'(0)\,\alpha^{-4} \quad \text{along the lower branch.} \quad (4.25)$$

5. The adjoint problem

The asymptotic theory of the adjoint Orr–Sommerfeld equation has recently been developed in some detail (Reid 1965). Although this theory of the adjoint equation has much in common with the usual theory of the Orr–Sommerfeld equation, some important differences do arise, particularly in connexion with the role played by the viscous corrections to the singular parts of the inviscid solutions. To examine some of these differences we shall therefore consider in this section the calculation of the curve of neutral stability for the asymptotic suction profile based on the adjoint formulation of the problem.

The adjoint problem for the asymptotic suction profile consists of the equation

$$(D^2-\alpha^2)^2\psi - (D^2-\alpha^2)D\psi = i\alpha R\{(D^2-\alpha^2)(U-c)\psi - U''\psi\} \quad (5.1)$$

together with the boundary conditions

$$\psi = \psi' = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad \psi, \psi' \to 0 \quad \text{as} \quad y \to +\infty. \quad (5.1a)$$

It is again necessary, therefore, to obtain approximations to the two solutions of equation (5.1) that remain bounded as $y \to +\infty$.

One of the required approximations can be obtained from the inviscid form of equation (5.1)

$$(D^2-\alpha^2)(U-c)\psi - U''\psi = 0, \tag{5.2}$$

the solutions of which can be written in the form

$$\psi_A(y) = Q_A(y-y_c) \tag{5.3}$$

and

$$\psi_B(y) = (y-y_c)^{-1}Q_B(y-y_c) + (U_c''/U_c')\,\psi_A(y)\log(y-y_c). \tag{5.4}$$

The functions Q_A and Q_B are power series in $y-y_c$, the leading coefficients of which are unity, and, to be definite, we shall suppose that ψ_B contains no multiple of ψ_A. If we now let $\Psi(y)$ denote the solution of equation (5.2) that remains bounded as $y \to +\infty$ and normalize it by the condition $\lim_{y\to y_c}\{(y-y_c)\Psi(y)\} = 1$, then we have

$$\Psi(y) = A\psi_A(y) + \psi_B(y), \tag{5.5}$$

where A is the same constant appearing in equation (3.3).

It will be noticed that ψ_A, like ϕ_A, is regular at the critical point, but that ψ_B has a stronger singularity there than ϕ_B. In fact, both terms in equation (5.4) are singular at the point $y = y_c$, the first like $(y-y_c)^{-1}$ and the second like $\log(y-y_c)$. Thus, to determine the correct branch of the multivalued solution ψ_B, it is necessary to consider at least two so-called 'viscous corrections'.

To obtain approximations to the viscous solutions of equation (5.1) and the viscous corrections to ψ_B, we first let

$$\psi(y) = \mathrm{X}(\xi), \quad \text{where} \quad \xi = (y-y_c)/\epsilon \quad \text{and} \quad \epsilon = (i\alpha R U_c')^{-\frac{1}{3}}. \tag{5.6}$$

Then $\mathrm{X}(\xi)$ satisfies the equation

$$\left(\frac{d^2}{d\xi^2}-\alpha^2\epsilon^2\right)^2\mathrm{X} - \epsilon\left(\frac{d^2}{d\xi^2}-\alpha^2\epsilon^2\right)\frac{d}{d\xi}\mathrm{X} = i\alpha R\epsilon^2\left\{\left(\frac{d^2}{d\xi^2}-\alpha^2\epsilon^2\right)(U-c)\,\mathrm{X} - \epsilon^2 U''\mathrm{X}\right\}. \tag{5.7}$$

If the solution of this equation is now expanded in the form

$$\mathrm{X}(\xi,\epsilon) = \mathrm{X}^{(0)}(\xi) + \epsilon(U_c''/U_c')\,\mathrm{X}^{(1)}(\xi) + \ldots, \tag{5.8}$$

then $\mathrm{X}^{(0)}(\xi)$ and $\mathrm{X}^{(1)}(\xi)$ satisfy the equations

$$\frac{d^2}{d\xi^2}\left(\frac{d^2}{d\xi^2}-\xi\right)\mathrm{X}^{(0)} = 0 \tag{5.9}$$

and

$$\frac{d^2}{d\xi^2}\left(\frac{d^2}{d\xi^2}-\xi\right)\mathrm{X}^{(1)} = \left(\frac{U_c'}{U_c''}\frac{d^3}{d\xi^3} + \frac{1}{2}\frac{d^2}{d\xi^2}\xi^2 - 1\right)\mathrm{X}^{(0)}. \tag{5.10}$$

The solutions of equation (5.9) can be taken in the form

$$\left.\begin{aligned} &\mathrm{X}_1^{(0)}(\xi) = 1, \quad \mathrm{X}_2^{(0)}(\xi) = \epsilon^{-1}Q_3(\xi), \\ &\mathrm{X}_3^{(0)}(\xi) = (\epsilon U_c')^{-\frac{1}{2}}P_1(\xi) \quad \text{and} \quad \mathrm{X}_4^{(0)}(\xi) = (\epsilon U_c')^{-\frac{1}{2}}P_2(\xi), \end{aligned}\right\} \tag{5.11}$$

where $P_1(\xi) = \mathrm{Ai}(\xi)$ and $P_2(\xi) = \mathrm{Ai}(\xi e^{\frac{2}{3}\pi i})$ are Airy functions. The function $Q_3(\xi)$ is defined by

$$Q_3(\xi) = 2\pi e^{\frac{1}{6}\pi i}\left\{P_1(\xi)\int_{\infty_2}^{\xi} P_2(\xi)\,d\xi - P_2(\xi)\int_{\infty_1}^{\xi} P_1(\xi)\,d\xi\right\}, \tag{5.12}$$

where the lower limits of integration (∞_1 or ∞_2) denote paths of integration that tend to infinity in the sectors S_1 ($|\arg\xi| < \frac{1}{3}\pi$) or $S_2(-\pi < \arg\xi < -\frac{1}{3}\pi)$ of the ξ-plane respectively.

The solution $X_1^{(0)}$ must clearly be identified with the leading term in the expansion (5.3) of the regular inviscid solution ψ_A. The solution $X_2^{(0)}$, however, provides one of the required viscous corrections to ψ_B. This can easily be seen from the fact that for $|\xi| \gg 1$, $X_2^{(0)}(\xi) \sim (\epsilon\xi)^{-1}$ in the sector $-\pi < \arg\xi < \frac{1}{3}\pi$ and is exponentially large in the sector $\frac{1}{3}\pi < \arg\xi < \pi$; it must therefore be identified, in some sense, with the term $(y-y_c)^{-1}$ in ψ_B. At this stage, it would be tempting, formally following Tollmien (1929; see also, Lin 1955, p. 128), to write down an 'improved' approximation to ψ_B in the form

$$\psi_B(y) = X_2^{(0)}(\xi)\,[Q_B(y-y_c) + (U_c''/U_c')\,\psi_A(y)\,(y-y_c)\log(y-y_c)], \tag{5.13}$$

where the remaining singularity in this solution is then as weak as

$$(y-y_c)\log(y-y_c),$$

i.e. as weak as the singularity in the uncorrected form of ϕ_B. This form for ψ_B, if it were correct, would have the advantage of leaving the viscous and inviscid parts of the characteristic equation completely separated, and thereby permitting a direct solution by means of the usual graphical construction. Calculations based on this form of ψ_B, however, led to totally unrealistic results, far worse, in fact, than when the uncorrected form of ψ_B was used. It would appear, therefore, that equation (5.13) does not provide a valid approximation to ψ_B. What, then, is the correct interpretation of the viscous solutions $X_1^{(0)}, X_2^{(0)}, \ldots$?

On closer examination of the expansion (5.8), which is actually a convergent expansion in ϵ for finite values of y, it would appear that these viscous solutions can provide only *term-by-term* improvements to ψ_B. Thus, to this order, we are led to consider the approximation

$$\psi_B(y) = X_2^{(0)}(\xi) + \frac{1}{y-y_c}[Q_B(y-y_c)-1] + \frac{U_c''}{U_c'}\psi_A(y)\log(y-y_c) \tag{5.14}$$

obtained by simply replacing the term $(y-y_c)^{-1}$ in ψ_B by $X_2^{(0)}$. The remaining singularity in this approximation is then only as weak as $\log(y-y_c)$, i.e. stronger than the singularity in the uncorrected form of ϕ_B. The calculations described below, based on this approximation to ψ_B, strongly suggest, however, that this approximation is completely adequate for most purposes.

To determime the correct branch of the logarithm appearing in ψ_B, it is necessary to consider the second viscous correction $X_2^{(1)}$. This aspect of the theory has been discussed in detail by Reid (1965) for the adjoint Orr–Sommerfeld equation and need not be repeated here. One remark should be made, however, in this connexion. In the present discussion of the stability of the asymptotic suction profile explicit account has been taken of the fact that the basic flow is not strictly parallel. As a result, equation (5.10) contains the additional term $d^3X^{(0)}/d\xi^3$ which is absent in the case of strictly parallel flows. Fortunately, however, this term does not affect the leading term in the asymptotic expansion of $X_2^{(1)}$ and so does not affect the determination of the correct branch of the logarithm in ψ_B. It does, of course, affect the behaviour of $X_2^{(1)}$ for finite values of ξ, and this fact shows that there is an intrinsic limitation in the usual stability analyses for nearly parallel flows of the boundary-layer type. A similar remark applies to the first viscous correction to ϕ_B.

Finally, it may be noted that the viscous solutions $X_3^{(0)}$ and $X_4^{(0)}$ are solutions of a second-order equation and thus no quadratures are required as in the corresponding viscous approximations to the Orr–Sommerfeld equation.

The solution of the adjoint characteristic equation

Having obtained suitable approximations to four solutions of the adjoint equation, we turn now to consider the solution of the characteristic equation for the adjoint problem. For this purpose it is convenient to define a complex parameter $w^\dagger(\alpha, c)$ in terms of $\psi(y)$ in exactly the same way that $w(\alpha, c)$ was defined in terms of $\Phi(y)$ [cf. equation (4.2)]. Thus we let

$$w^\dagger(\alpha, c) = \left[1 + \frac{U'(0)}{c}\,\frac{\Psi(0)}{\Psi'(0)}\right]^{-1}. \tag{5.15}$$

Since Ψ is proportional to $\Phi/(U-c)$, w and $w^\dagger$ are related by

$$(w^\dagger - 1)/w^\dagger = w - 1. \tag{5.16}$$

As a first approximation to ψ, consider a linear combination of $\Psi(y)$ and $\mathrm{Ai}\,(\xi)$, where Ψ is taken as a linear combination of ψ_A and the uncorrected form of ψ_B. In this approximation, the characteristic equation can be written in the form [cf. equation (4.1)]

$$(w^\dagger - 1)/(1+\lambda)\, w^\dagger = F^\dagger(z), \tag{5.17}$$

where $F^\dagger(z)$ is the adjoint Tietjens function defined by

$$F^\dagger(z) = \mathrm{Ai}\,(\xi_1)/\xi_1\, \mathrm{Ai}'(\xi_1) \tag{5.18}$$

and $\xi_1 = z e^{-\frac{5}{6}\pi i}$ with z real in the neutral case. Like the Tietjens function, $F^\dagger(z)$ is a universal function, independent of the basic velocity profile. Tables of $F^\dagger(z)$ and $F^{\dagger\prime}(z)$ have been given by Reid (1965) for $z = 0{\cdot}1\ (0{\cdot}1)\ 2\ (0{\cdot}5)\ 10$; $4S$.

The curve of neutral stability based on equation (5.17) can now be obtained in a direct manner by means of the graphical construction described in § 4 and shown in figure 5. The left-hand side of equation (5.17) was computed by using the numerical procedure outlined in § 3 and the relationship (5.16) between w and $w^\dagger$. The curve of neutral stability obtained in this way is shown as the long-dashed curve in figure 2.

Since $\alpha \to 0$ as $R \to \infty$ along both branches of the neutral curve, we can use the small α approximation to w obtained in § 4 to derive the asymptotes to the neutral curve. From equations (4.11) and (5.16), and using the fact that $U'(0) = 1$, we have

$$\frac{c}{1+\lambda}\left[\frac{1}{\alpha(1-c)^2} + \Omega_0(0) + O(\alpha)\right] = F^\dagger(z), \tag{5.19}$$

where, for small values of c,

$$\left.\begin{aligned} \mathrm{Re}\,\{\Omega_0(0)\} = -1/c + O(\log c),\ \mathrm{Im}\,\{\Omega_0(0)\} &= \pi/(1-c)^2 \\ \text{and} \qquad 1+\lambda &= 1 + \tfrac{1}{2}c + O(c^2). \end{aligned}\right\} \tag{5.20}$$

From these results it follows that

$$F_r^\dagger(z) \to c/\alpha - 1 \quad \text{as} \quad \alpha \to 0. \tag{5.21}$$

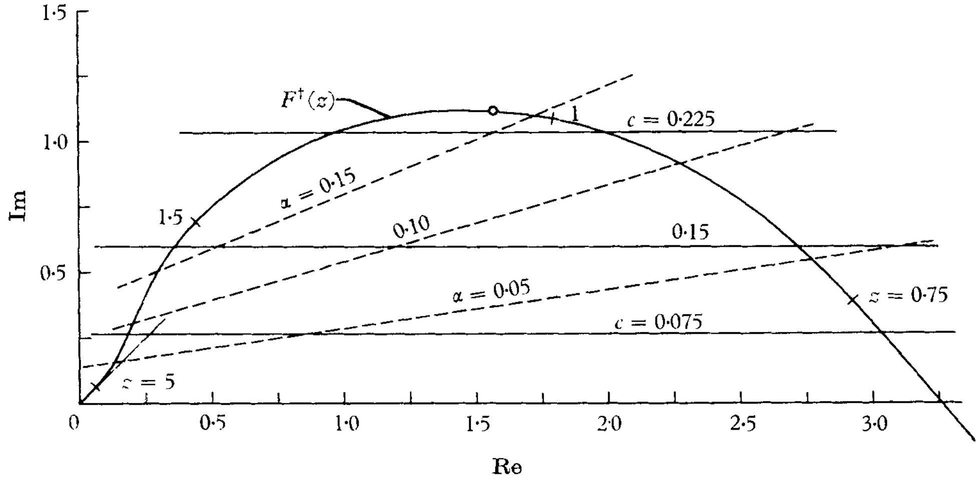

FIGURE 5. The graphical solution of the adjoint characteristic equation (5.17). The circled point corresponds to the minimum critical Reynolds number.

Since z tends to a finite value z_0 (say) as $R \to \infty$ along the lower branch and $F^{\dagger}(z) \to 0$ as $z \to \infty$ along the upper branch (cf. figure 5), we have

$$c \sim [1 + F_r^{\dagger}(z_0)]\,\alpha \quad \text{and} \quad c \sim \alpha \tag{5.22}$$

along the lower and upper branches, respectively. Equation (5.18) shows that $F_i^{\dagger}(z) \to 0$ like c along the lower branch so that z_0 is defined by the conditions $F_i^{\dagger}(z_0) = 0$ with $F_r^{\dagger}(z_0) \neq 0$. From the tables of $F^{\dagger}(z)$ we obtain $z_0 \fallingdotseq 0{\cdot}661$ and $F_r^{\dagger}(z_0) \fallingdotseq 3{\cdot}253$. Since $z = (\alpha R U_c')^{\frac{1}{3}} y_c$, we see that the lower asymptote to the neutral curve is given by the equation $R^{\frac{1}{3}} = 0{\cdot}1565\alpha^{-\frac{4}{3}}$. Along the upper branch $z \to \infty$ and since the leading terms in the asymptotic expansions of $F(z)$ and $F^{\dagger}(z)$ are the same, it follows that the upper asymptote is the same as the one given by equation (4.24).

Because the singularity in ψ_B is stronger than the singularity in ϕ_B, the lack of agreement between these two neutral curves is not entirely unexpected. In a consistent approximation it is clear that one must include the viscous solution $X_2^{(0)}(\xi)$. As already mentioned, the 'improved' approximation to ψ_B given by equation (5.13) is not satisfactory and, for a second approximation to ψ, we therefore take

$$\psi(y) = \Psi(y) - (y - y_c)^{-1} + \epsilon^{-1} Q_3(\xi) + \text{constant}\,\mathrm{Ai}(\xi), \tag{5.23}$$

where $\Psi(y)$ still denotes the uncorrected inviscid solution (5.5). In this approximation the characteristic equation can be written in the form

$$\frac{-y_c \Psi(0) - 1 + \xi_1 Q_3(\xi_1)}{y_c^2 \Psi'(0) + 1 + \xi_1^2 Q_3'(\xi_1)} = F^{\dagger}(z), \tag{5.24}$$

where $\xi_1 = z e^{-\frac{5}{6}\pi i}$ as before.

Since it is not possible to write equation (5.24) in a form that separates the viscous and inviscid terms, the usual graphical method of solution cannot be used. Equation (5.24) can be solved, however, by first choosing a value of z and then plotting the real and imaginary parts of both sides of the equation, the left-hand side being evaluated for various values of α and c. An example of such a plot is shown in figure 6 for $z = 2{\cdot}7$. The necessary tables of $Q_3(\xi_1)$ and $Q_3'(\xi_1)$ have been given by Reid (1965) for $z = 0(0{\cdot}2)10$; 4*S*. In this way, the curve

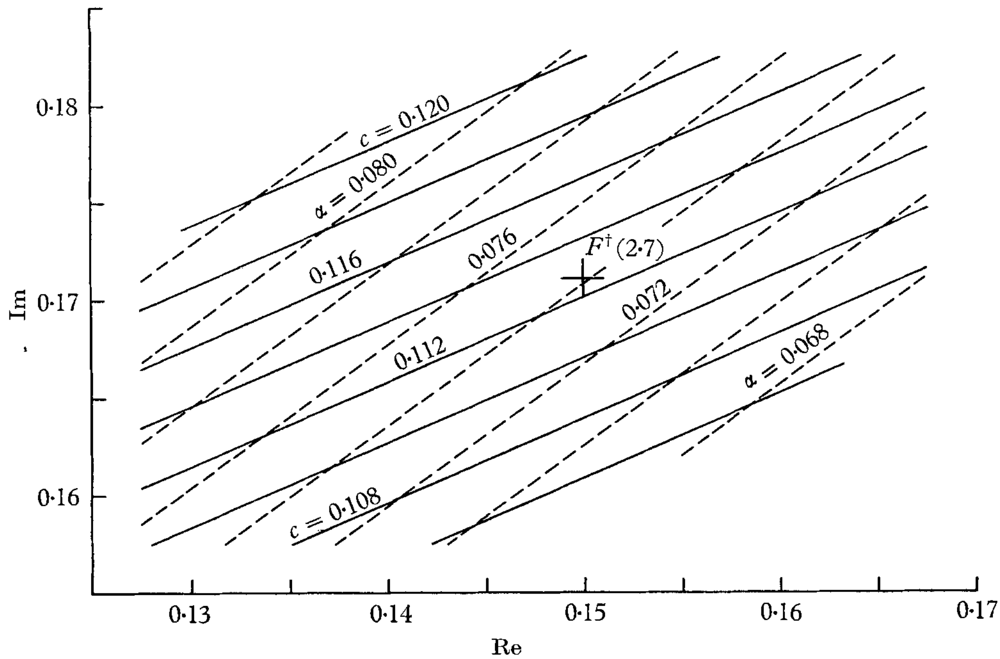

FIGURE 6. An example of the graphical solution of the adjoint characteristic equation (5.24) for $z = 2{\cdot}7$.

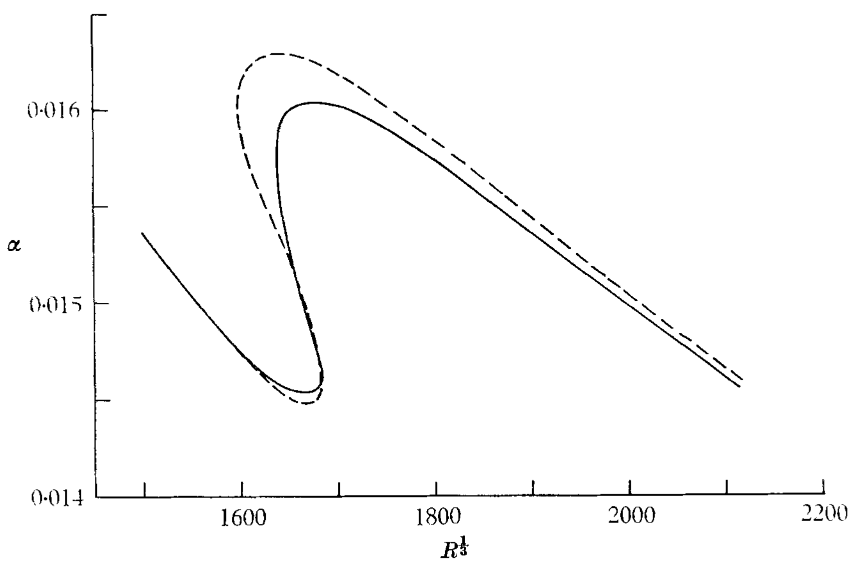

FIGURE 7. The kinks in the neutral curves based on equation (4.1) (solid curve) and equation (5.24) (dashed curve).

of neutral stability can be found, point by point, by computing and plotting a new graph like figure 6 for each value of z.

To avoid the need for such a graphical construction for each value of z, which is a tedious procedure at best, a method was devised that uses a high speed computer to solve directly a characteristic equation of the form (5.24). Having fixed the value of z, an estimate is first made of the values of α and c that will solve the characteristic equation. Next we compute the length $\delta = |F^{\dagger}(z) - G(\alpha, c, z)|$, where $G(\alpha, c, z)$ denotes the left-hand side of equation (5.24). The parameters α and c are then varied until δ is as small as desired. In this way it is easy to obtain as many points as may be required on the neutral curve. The results of such calculations, based on the characteristic equation (5.24), are shown in figures 2, 3, and 7.

The excellent agreement between the curves of neutral stability computed from equations (4.1) and (5.24) is further emphasized by the fact that both the upper and lower asymptotes are virtually identical. Along the lower branch of the neutral curve, α and c both tend to zero and in this limit equation (5.24) becomes

$$[-y_c A(\alpha) + \xi_1 Q_3(\xi_1)]/\xi_1^2 Q_3'(\xi_1) = F^{\dagger}(z). \tag{5.25}$$

Since $A(\alpha) \sim 1/\alpha$ as $\alpha \to 0$ and $y_c \sim c$ as $c \to 0$, the limiting form of this equation becomes

$$c/\alpha = \xi_1 Q_3(\xi_1) - \xi_1^2 Q_3'(\xi_1) F^{\dagger}(z). \tag{5.26}$$

If we denote the right-hand side of this equation by $Q(z)$, then in this limit there must exist a value of z, z_0' say, such that

$$c/\alpha = Q_r(z_0') \quad \text{and} \quad Q_i(z_0') = 0. \tag{5.27}$$

From the tables of $F^{\dagger}(z)$ and $Q_3(\xi_1)$ we find that $z_0' \fallingdotseq 2{\cdot}2972$ and $Q_i(z_0') \fallingdotseq 2{\cdot}2957$, and hence that $R^{\frac{1}{3}} \sim 1{\cdot}001\alpha^{-\frac{4}{3}}$. To the present accuracy, this result is identical with equation (4.25). Along the upper branch, $z \to \infty$ and equation (5.24) then has the limiting form

$$-\Psi(0)/y_c \Psi'(0) = F^{\dagger}(z), \tag{5.28}$$

which, by equation (5.13), can be shown to be identical with equation (5.15). The upper asymptote must therefore be the same as equation (4.24).

6. Concluding remarks

The exact solution of the inviscid equation presented in §2 is of importance largely because of the insight it provides into the structure of the inviscid solutions. It shows, for example, the important role played by analytical continuation in the case of flows of the boundary-layer type. More generally, the inviscid solutions would have to be found by the numerical procedure described in §3, or some modification of it. When this numerical method of solving the inviscid equation is combined with the fully automated methods of solving the characteristic equation described in §5, the calculation of a curve of neutral stability becomes almost a routine matter.

The present discussion has been limited to the case of neutral stability for which c is real. But it is clear that this exact solution of the inviscid equation could also be advantageously used to study a number of other aspects of the stability problem such as the inviscid initial value problem, the higher, damped modes for the viscous problem, or the spatial stability problem in which αc is real but α may be complex.

A point of some theoretical interest that emerges from this work concerns the kink on the upper branch of the neutral curves shown in figures 2 and 7. Based on the usual asymptotic approximations to the solutions of the Orr–Sommerfeld equation, we have the characteristic equation (4.1) and, in this approximation, the kink can clearly be traced to the loop in the Tietjens function shown in figure 1. This is only a very superficial explanation, however; more fundamentally, the existence of such a kink, which would seem to be without any physical significance, would strongly suggest some defect in the viscous approximations being used. When the kink was first discovered—and this was long before the asymptotic theory for the adjoint equation had been developed—it was natural to look for an explanation of it in terms of the viscous correction to ϕ_B. At that time, even this explanation was not wholly convincing in view of the fact that the kink occurs at such a large value of z. Once the calculations described in §5 had been completed, however, it became immediately clear that in the adjoint formulation of the problem at least the first viscous correction must be included to obtain an adequate approximation and that this viscous correction is actually responsible for the appearance of the kink. It is thus reasonable to suppose that if the first viscous correction to ϕ_B or the first two viscous corrections to ψ_B were included, then the kink might again disappear.

In this connexion it is necessary to consider precisely how the viscous corrections should be applied. In the method first suggested by Tollmien (1929), the singularity in ϕ_B is removed by introducing a viscous function in such a way that the corrected form of ϕ_B is regular at the critical point. Whereas, according to the present point of view (see also Reid 1965 for a more detailed discussion), the first viscous correction to ϕ_B should only be used to replace the term

$$(y-y_c)\log(y-y_c)$$

in the series representation of ϕ_B, leaving thereby a weaker singularity of order $(y-y_c)^2\log(y-y_c)$.

Although Holstein (1950, see also Stuart 1963) has given tables of the first viscous correction to ϕ_B based on the Orr–Sommerfeld equation, no attempts have previously been made to include this viscous correction in the calculation of a neutral curve because of the difficulty in solving the resulting characteristic equation. Since this last difficulty has now been overcome by the method described in §5, calculations were made of the neutral curve for the asymptotic suction profile based on both methods of applying the viscous correction to ϕ_B. Unfortunately, because of the limited accuracy of Holstein's tables, the results were inconclusive. They suggest, however, that the differences which result from the two methods of applying the viscous correction are not large and that a discussion of the precise behaviour of the neutral curve in the neighbourhood of the kink must await the calculation of more accurate tables of the viscous correction to ϕ_B

This work has been supported in part by the National Science Foundation under grant No. GP-2246. It was completed while one of us (W. H. R.) was a Fulbright Research Scholar in the Department of Mathematics, Institute of Advanced Studies, The Australian National University, Canberra.

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